

On the property (Z_{E_a})

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Abstract

The paper introduces the notion of properties (Z_{Π_a}) and (Z_{E_a}) as variants of Weyl's theorem and Browder's theorem for bounded linear operators acting on infinite dimensional Banach spaces. A characterization of these properties in terms of localized single valued extension property is given, and the perturbation by commuting Riesz operators is also studied. Classes of operators are considered as illustrating examples.

1 Introduction

In 1909 H.Weyl [19] examined the spectra of all compact perturbation of a self-adjoint operator on a Hilbert space and found that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. Today this classical result may be stated by saying that the spectral points of a self-adjoint operator which do not belong to Weyl spectrum are precisely the eigenvalues of finite multiplicity which are isolated points of the spectrum. This Weyl's theorem has been extended from self-adjoint operators to several other classes of operators and many new variants have been obtained by many researchers ([6], [7], [8], [10], [13], [16]).

This paper is a continuation of our recent investigations in the subject of Weyl type theorems. We introduce and study the new variants of Weyl's theorem and Browder's theorem. The essential results obtained are summarized in the diagram presented in the end of the second section of this paper. For further definitions and symbols we also refer the reader to [7], [8] and [20].

We begin with some standard notations of Fredholm theory. Throughout this paper let $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex Banach space X . For an operator $T \in \mathcal{B}(X)$, we denote by T^* , $\sigma(T)$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ the dual of T , the spectrum of T , the null space of T and the range space of T , respectively. If $\dim \mathcal{N}(T) < \infty$ and $\dim \mathcal{N}(T^*) < \infty$, then T is called a *Fredholm* operator and its index is defined by $\text{ind}(T) = \dim \mathcal{N}(T) - \dim \mathcal{N}(T^*)$. A *Weyl* operator is a Fredholm operator of index 0 and the Weyl spectrum is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\}$.

For a bounded linear operator T and $n \in \mathbb{N}$, let $T_{[n]} : \mathcal{R}(T^n) \rightarrow \mathcal{R}(T^n)$ be the restriction of T to $\mathcal{R}(T^n)$. $T \in L(X)$ is said to be *B-Weyl* if for some integer $n \geq 0$ the range $\mathcal{R}(T^n)$ is closed and

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$T_{[n]}$ is Weyl; its index is defined as the index of the Weyl operator $T_{[n]}$. The respective *B-Weyl spectrum* is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}$, see [5].

The *ascent* $a(T)$ of an operator T is defined by $a(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$, and the *descent* $\delta(T)$ of T is defined by $\delta(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$, with $\inf \emptyset = \infty$. An operator $T \in \mathcal{B}(X)$ is called Browder if it is Fredholm of finite ascent, and finite descent and the respective Browder spectrum is defined by $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Browder operator}\}$. According to [12], a complex number $\lambda \in \sigma(T)$ is a *pole* of the resolvent of T if $T - \lambda I$ has finite ascent and finite descent, and in this case they are equal. We recall [7] that a complex number $\lambda \in \sigma_a(T)$ is a *left pole* of T if $a(T - \lambda I) < \infty$ and $\mathcal{R}(T^{a(T - \lambda I) + 1})$ is closed. In addition, we have the following usual notations that will be needed later:

Notations and symbols:

$\mathcal{F}(X)$: the ideal of finite rank operators in $\mathcal{B}(X)$,

$\mathcal{K}(X)$: the ideal of compact operators in $\mathcal{B}(X)$,

$\mathcal{N}(X)$: the class of nilpotent operators on X ,

$\mathcal{Q}(X)$: the class of quasi-nilpotent operators on X ,

$\mathcal{R}(X)$: the class of Riesz operators acting on X ,

iso A : isolated points of a subset $A \subset \mathbb{C}$,

acc A : accumulations points of a subset $A \subset \mathbb{C}$,

$D(0, 1)$: the closed unit disc in \mathbb{C} ,

$C(0, 1)$: the unit circle of \mathbb{C} ,

$\Pi(T)$: poles of T ,

$\Pi^0(T)$: poles of T of finite rank,

$\Pi_a(T)$: left poles of T ,

$\sigma_p(T)$: eigenvalues of T ,

$\sigma_p^0(T)$: eigenvalues of T of finite multiplicity,

$E^0(T) := \text{iso } \sigma(T) \cap \sigma_p^0(T)$,

$E(T) := \text{iso } \sigma(T) \cap \sigma_p(T)$,

$E_a(T) := \text{iso } \sigma_a(T) \cap \sigma_p(T)$,

$\sigma_b(T) = \sigma(T) \setminus \Pi^0(T)$: Browder spectrum of T ,

$\sigma_W(T)$: Weyl spectrum of T ,

$\sigma_{BW}(T)$: B-Weyl spectrum of T ,

the symbol \sqcup stands for the disjoint union.

Definition 1.1. [7], [11], [19] Let $T \in \mathcal{B}(X)$. T is said to satisfy

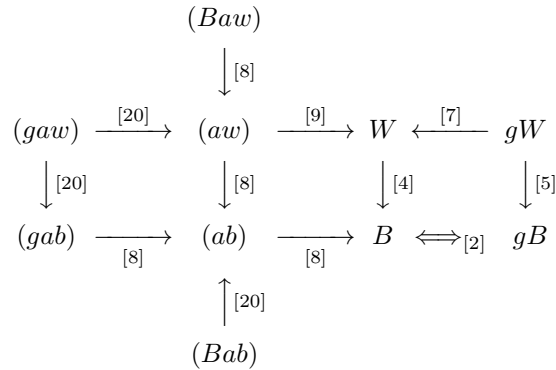
- i) Weyl's theorem if $\sigma(T) \setminus \sigma_W(T) = E^0(T)$; (W for brevity).
- ii) Browder's theorem if $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$; (B for brevity).
- iii) generalized Weyl's theorem if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$; (gW for brevity).
- iv) generalized Browder's theorem if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$; (gB for brevity).

Definition 1.2. [8],[20] Let $T \in \mathcal{B}(X)$. T is said to satisfy

- i) Property (gab) if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$.

- ii) Property (gaw) if $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$.
- iii) Property (ab) if $\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$.
- iv) Property (aw) if $\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$.
- v) Property (Bab) if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$.
- vi) Property (Baw) if $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$.

The relationship between properties and theorems given in the precedent definitions is summarized in the following diagram. (arrows signify implications and numbers near the arrows are references to the bibliography therein).



Moreover, counterexamples were given to show that the reverse of each implication in the diagram is not true. Nonetheless, it was proved that under some extra assumptions, these implications are equivalences.

2 Properties (Z_{Π_a}) and (Z_{E_a})

We define the properties (Z_{Π_a}) and (Z_{E_a}) as follows:

Definition 2.1. A bounded linear operator $T \in \mathcal{B}(X)$ is said to satisfy property (Z_{E_a}) if $\sigma(T) \setminus \sigma_W(T) = E_a(T)$, and is said to satisfy property (Z_{Π_a}) if $\sigma(T) \setminus \sigma_W(T) = \Pi_a(T)$.

Example 2.2. Hereafter, we denote by R the unilateral right shift operator defined on the $\ell^2(\mathbb{N})$ by $R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$.

1. It is well known that $\sigma(R) = D(0, 1)$, $\sigma_W(R) = D(0, 1)$ and $E_a(R) = \Pi_a(R) = \emptyset$. So R satisfies the property (Z_{E_a}) and the property (Z_{Π_a}) .
2. Let V denote the Volterra operator on the Banach space $C[0, 1]$ defined by $V(f)(x) = \int_0^x f(t)dt$ for all $f \in C[0, 1]$. V is injective and quasinilpotent. $\sigma(V) = \sigma_W(V) = \{0\}$ and $\Pi_a(V) = E_a(V) = \emptyset$. So V satisfies the properties (Z_{E_a}) and (Z_{Π_a}) .

Lemma 2.3. Let $T \in \mathcal{B}(X)$. If T satisfies property (Z_{E_a}) , then

$$E_a(T) = E_a^0(T) = \Pi_a^0(T) = \Pi_a(T) = \Pi^0(T) = \Pi(T) = E^0(T) = E(T).$$

Proof. Suppose that T satisfies property (Z_{E_a}) , then $\sigma(T) = \sigma_W(T) \sqcup E_a(T)$. Thus $\mu \in E_a(T) \iff \mu \in \text{iso } \sigma_a(T) \cap \sigma_W(T)^C \implies \mu \in \Pi_a^0(T)$, where $\sigma_W(T)^C$ is the complement of the Weyl spectrum of T . Hence $E_a(T) = E_a^0(T) = \Pi_a^0(T) = \Pi_a(T)$, $\Pi(T) = \Pi^0(T)$ and $E(T) = E^0(T)$. Consequently, $\sigma(T) = \sigma_W(T) \sqcup E_a^0(T)$. This implies that $E^0(T) = \Pi^0(T)$. Hence $E_a(T) = E_a^0(T) = \Pi_a^0(T) = \Pi_a(T)$ and $\Pi^0(T) = \Pi(T) = E^0(T) = E(T)$. Since the inclusion $\Pi(T) \subset \Pi_a(T)$ is always true, it suffices to show its opposite. If $\mu \in \Pi_a(T)$, then $a(T - \mu I)$ is finite and since T satisfies property (Z_{E_a}) , it follows that $\mu \in \Pi(T)$ and hence the equality desired. \square

Corollary 2.4. *Let $T \in \mathcal{B}(X)$. The following statements are equivalent:*

- i) T satisfies property (Z_{E_a}) ;
- ii) T satisfies Weyl's theorem and $E^0(T) = E_a(T)$;
- iii) T satisfies Browder's theorem and $\Pi^0(T) = E_a(T)$.
- iv) T satisfies generalized Weyl's Theorem and $E^0(T) = E_a(T)$;

Proof. The equivalence between the first three statements is clear.

To prove the equivalence between (i) and (iv). If T satisfies property (Z_{E_a}) , then T satisfies Browder's theorem and then generalized Browder's theorem too. Thus from Lemma 2.3, T satisfies generalized Weyl's theorem and $E^0(T) = E_a(T)$. Conversely, suppose that T satisfies generalized Weyl's and $E^0(T) = E_a(T)$. From [7, Theorem 3.9], T satisfies Weyl's theorem $\sigma(T) \setminus \sigma_W(T) = E^0(T) = E_a(T)$. So T satisfies property (Z_{E_a}) . \square

Following [13], an operator $T \in \mathcal{B}(X)$ is said to satisfy property (k) if $\sigma(T) \setminus \sigma_W(T) = E(T)$. For the definition of property (k) , see also the reference [6] in which this property is named (W_E) . From Lemma 2.3 we have immediately the next corollary:

Corollary 2.5. *Let $T \in \mathcal{B}(X)$. The following statements are equivalent:*

- i) T satisfies property (Z_{E_a}) ;
- ii) T satisfies property (Z_{Π_a}) and $E_a(T) = \Pi_a(T)$;
- iii) T satisfies property (k) and $E_a(T) = E(T)$.

Example 2.6. Generally, we cannot expect that property (Z_{E_a}) holds for an operator satisfying property (Z_{Π_a}) or property (k) , as we can see in the following example.

1. Let $Q \in \mathcal{B}(X)$ be a quasi-nilpotent operator acting on an infinite dimensional Banach space X such that $\mathcal{R}(Q^n)$ is non-closed for all $n \in \mathbb{N}$ and let $T = 0 \oplus Q$ defined on the Banach space $X \oplus X$. Clearly, $\sigma_W(T) = \sigma_{BW}(T) = \sigma(T) = \{0\}$, $E_a(T) = \{0\}$ and $\Pi_a(T) = \emptyset$. So T satisfies property (Z_{Π_a}) , but it does not satisfy property (Z_{E_a}) .
2. Let T be the operator given by the direct sum of the unilateral right shift operator R and the projection operator U defined in the first point of Remark 2.8 below. Then $\sigma(T) = D(0, 1)$, $\sigma_W(T) = D(0, 1)$, $E(T) = \emptyset$. So T satisfies property (k) , but it does not satisfy property (Z_{E_a}) , since $E_a(T) = \{0\}$.

In the following theorem we establish a relationship between property (Z_{E_a}) and the properties (gaw) , (aw) , (Baw) (see Definition 1.2).

Theorem 2.7. *Let $T \in \mathcal{B}(X)$. The following statements are equivalent:*

- i) T satisfies property (Z_{E_a}) ;
- ii) T satisfies property (gaw) and $\sigma_{BW}(T) = \sigma_W(T)$;
- iii) T satisfies property (aw) and $E_a(T) = E_a^0(T)$;
- iv) T satisfies property (Baw) and $E_a(T) = E_a^0(T)$;

Proof. (i) \iff (iii) Suppose that T satisfies property (Z_{E_a}) , then from Lemma 2.3, $\sigma(T) = \sigma_W(T) \sqcup E_a(T)$ and $E_a(T) = \sigma_W(T) \sqcup E_a^0(T)$. So T satisfies property (aw) and $E_a(T) = E_a^0(T)$. The converse is clear.
 (i) \iff (ii) If T satisfies property (Z_{E_a}) , then it satisfies property (aw) . Since by Lemma 2.3 we have $E_a(T) = \Pi(T)$, it follows from [9, Theorem 2.2] that T satisfies property (gaw) , and this entails that $\sigma_{BW}(T) = \sigma(T) \setminus E_a(T) = \sigma_W(T)$. The converse is obvious.
 (i) \iff (iv) If T satisfies property (Z_{E_a}) , then $\sigma_{BW}(T) = \sigma_W(T)$ and $E_a(T) = E_a^0(T)$. So $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$, i.e. T satisfies property (Baw) . Conversely, suppose that T satisfies property (Baw) and $E_a(T) = E_a^0(T)$. By [20, Theorem 3.3] we have $\sigma_{BW}(T) = \sigma_W(T)$. Thus $E_a(T) = E_a^0(T) = \sigma(T) \setminus \sigma_{BW}(T) = \sigma(T) \setminus \sigma_W(T)$, and T satisfies property (Z_{E_a}) . \square

Remark 2.8. From Theorem 2.7, if $T \in \mathcal{B}(X)$ satisfies property (Z_{E_a}) then it satisfies property (δ) ; where $\delta \in \{gaw, aw, Baw\}$. However, the converse in general is not true. To see this,

1. Let $U \in L(\ell^2(\mathbb{N}))$ be defined by $U(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$. Then $\sigma(U) = \{0, 1\}$, $\sigma_W(U) = \{1\}$, $E_a(U) = \{0, 1\}$ and $\sigma_{BW}(U) = \emptyset$. So U satisfies property (gaw) and then property (aw) . But it does not satisfy property (Z_{E_a}) , because $\sigma(U) \setminus \sigma_W(U) \neq E_a(U)$. Here $E_a^0(U) = \{0\}$.
2. On the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, we consider the operator T defined by $T = 0 \oplus R$. We have T satisfies property (Baw) , since $\sigma(T) = \sigma_{BW}(T) = D(0, 1)$ and $E_a^0(T) = \emptyset$. But it does not satisfy property (Z_{E_a}) , since $\sigma_W(T) = D(0, 1)$ and $E_a(T) = \{0\}$.

Lemma 2.9. Let $T \in \mathcal{B}(X)$. If T satisfies property (Z_{Π_a}) , then

$$\Pi_a^0(T) = \Pi_a(T) = \Pi^0(T) = \Pi(T).$$

Proof. Suppose that T satisfies property (Z_{Π_a}) , that's $\sigma(T) = \sigma_W(T) \sqcup \Pi_a(T)$. Then $\mu \in \Pi_a(T) \iff \mu \in \text{iso } \sigma_a(T) \cap \sigma_W(T)^C \implies \mu \in \Pi_a^0(T)$. This implies that $\Pi_a(T) = \Pi_a^0(T)$ and $\Pi(T) = \Pi^0(T)$. So $\sigma(T) = \sigma_W(T) \sqcup \Pi_a^0(T)$ and this implies that $\Pi^0(T) = \Pi_a^0(T)$. Therefore $\Pi(T) = \Pi^0(T) = \Pi_a(T) = \Pi_a^0(T)$. \square

In the following theorem we establish a relationship between the property (Z_{Π_a}) , the properties (gab) , (ab) , (Bab) and the classical Browder's theorem (see Definition 1.1).

Theorem 2.10. *Let $T \in \mathcal{B}(X)$. Then the following statements are equivalent:*

- i) T satisfies property (Z_{Π_a}) ;

- ii) T satisfies property (gab) and $\sigma_{BW}(T) = \sigma_W(T)$;
- iii) T satisfies property (ab) and $\Pi_a(T) = \Pi_a^0(T)$;
- iv) T satisfies property (Bab) and $\Pi_a(T) = \Pi_a^0(T)$.
- v) T satisfies Browder's theorem and $\Pi_a(T) = \Pi^0(T)$.

Proof. (i) \iff (ii) Suppose that T satisfies property (Z_{Π_a}) , that's $\sigma(T) = \sigma_W(T) \sqcup \Pi_a(T)$. From Lemma 2.9, $\sigma(T) = \sigma_W(T) \sqcup \Pi_a^0(T)$. So T satisfies property (ab) . As $\Pi(T) = \Pi_a(T)$, then from [8, Theorem 2.8], T satisfies property (gab) . Moreover, $\sigma_{BW}(T) = \sigma(T) \setminus \Pi_a(T) = \sigma_W(T)$. The reverse implication is obvious.

(i) \iff (iii) Follows directly from Lemma 2.9.

(i) \iff (iv) If T satisfies property (Z_{Π_a}) , then $\sigma(T) \setminus \sigma_{BW}(T) = \sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T) = \Pi_a(T)$. So T satisfies property (Bab) . Conversely, the property (Bab) for T implies from [20, Theorem 3.6] that $\sigma_{BW}(T) = \sigma_W(T)$. So $\sigma_W(T) = \sigma(T) \setminus \Pi_a^0(T) = \sigma(T) \setminus \Pi_a(T)$ and this means that T satisfies property (Z_{Π_a}) . The equivalence between assertions (i) and (v) is clear. \square

Remark 2.11. From Theorem 2.10, It follows that:

1. If T satisfies property (Z_{Π_a}) , then it satisfies property (gab) and then property (ab) and Browder's theorem. But the converses are not true in general. For this, let $T \in L(\ell^2(\mathbb{N}))$ be defined by $T(x_1, x_2, x_3, \dots) = (0, 0, \frac{1}{3}x_1, 0, 0, \dots)$. Thus $\sigma(T) = \sigma_W(T) = \{0\}$ and $\Pi_a(T) = \{0\}$ and since T is nilpotent, then $\sigma_{BW}(T) = \emptyset$. So T satisfies property (gab) and then property (ab) and Browder's theorem. But T does not satisfy property (Z_{Π_a}) .
2. Also we cannot expect that property (Z_{Π_a}) holds for an operator satisfying property (Bab) , as we can see in the following: It is easily seen that the operator T defined in the second point of Remark 2.8 satisfies property (Bab) and it does not satisfy property (Z_{Π_a}) . Here $\Pi_a(T) = \{0\}$ and $\Pi_a^0(T) = \emptyset$.

The following property has relevant role in local spectral theory: a bounded linear operator $T \in \mathcal{B}(X)$ is said to have the *single-valued extension property* (SVEP for short) at $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ , the function $f \equiv 0$ is the only analytic solution of the equation $(T - \mu I)f(\mu) = 0 \quad \forall \mu \in U_\lambda$. We denote by $\mathcal{S}(T) = \{\lambda \in \mathbb{C} : T \text{ does not have SVEP at } \lambda\}$ and we say that T has SVEP if $\mathcal{S}(T) = \emptyset$. We say that T has SVEP on $A \subset \mathbb{C}$, if T has SVEP at every $\lambda \in A$. (For more details about this property, we refer the reader to [14]).

Proposition 2.12. *Let $T \in \mathcal{B}(X)$. If T or its dual T^* has SVEP on $\sigma_W(T)^C$ then T satisfies property (Z_{E_a}) if and only if $E_a(T) = \Pi^0(T)$; where $\sigma_W(T)^C$ is the complement of the Weyl spectrum of T .*

Proof. If T satisfies property (Z_{E_a}) , then from Lemma 2.3, $E_a(T) = \Pi^0(T)$. Remark that in this implication, the condition of SVEP for T or T^* is not necessary. Conversely, assume that $E_a(T) = \Pi^0(T)$. Note that T has SVEP on $\sigma_W(T)^C \iff T^*$ has SVEP on $\sigma_W(T)^C$. But this is equivalent to say that T satisfies Browder's theorem $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T) = E_a(T)$. So T satisfies property (Z_{E_a}) . \square

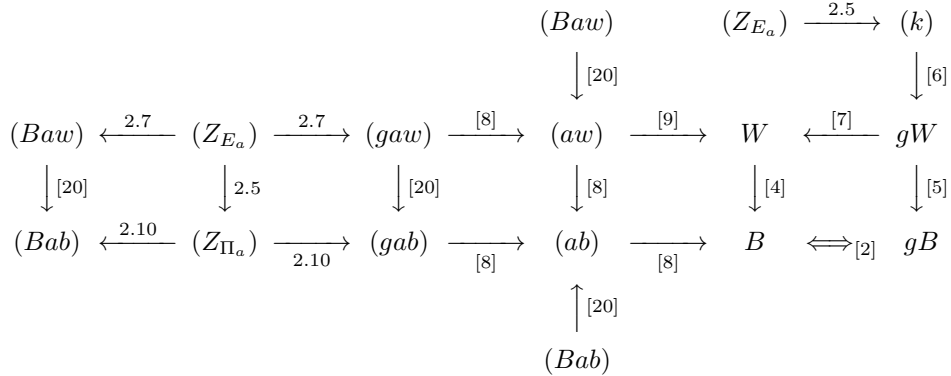
Remark 2.13. The hypothesis T or T^* has SVEP on $\sigma_W(T)^C$ is crucial as shown in this example: define the operator T by $T = R \oplus R^*$. We have $\sigma(T) = D(0, 1)$ and $\Pi^0(T) = \emptyset$. But, since $\dim \mathcal{N}(T) = \text{codim} \mathcal{R}(T) = 1$, then $0 \notin \sigma_W(T)$. So T does not satisfy property (Z_{E_a}) . Note that T and T^* do not have SVEP at $0 \in \sigma_W(T)^C$, as $\mathcal{S}(T) = \mathcal{S}(T^*) = \mathcal{S}(U^*) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| < 1\}$.

Similarly, we have the following proposition for the property (Z_{Π_a}) .

Proposition 2.14. *If $T \in \mathcal{B}(X)$ or its dual T^* has SVEP on $\sigma_W(T)^C$ then T satisfies property (Z_{Π_a}) if and only if $\Pi_a(T) = \Pi^0(T)$.*

Proof. Obtained by an argument similar to the one of the preceding proof. \square

Now, we give a summary of the results obtained in this section. In the following diagram which is a combination with the first presented above, arrows signify implications and the numbers near the arrows are references to the results obtained in in this section (numbers without brackets) or to the bibliography therein (the numbers in square brackets).



3 Preservation under commuting Riesz perturbations

We recall that an operator $R \in \mathcal{B}(X)$ is said to be *Riesz* if $R - \mu I$ is Fredholm for every non-zero complex μ , that is, $\pi(R)$ is quasinilpotent in the Calkin algebra $C(X) = \mathcal{B}(X)/\mathcal{K}(X)$ where π is the canonical mapping of $\mathcal{B}(X)$ into $C(X)$.

We denote by $\mathcal{F}^0(X)$, the class of finite rank power operators as follows:

$$\mathcal{F}^0(X) = \{S \in \mathcal{B}(X) : S^n \in \mathcal{F}(X) \text{ for some } n \in \mathbb{N}\}.$$

Clearly,

$$\mathcal{F}(X) \cup \mathcal{N}(X) \subset \mathcal{F}^0(X) \subset \mathcal{R}(X), \text{ and } \mathcal{K}(X) \cup \mathcal{Q}(X) \subset \mathcal{R}(X).$$

We start this section by the following nilpotent perturbation result.

Proposition 3.1. *Let $T \in \mathcal{B}(X)$ and let $N \in \mathcal{N}(X)$ which commutes with T . Then T satisfies property (s) if and only if $T + N$ satisfies property (s) ; where $s \in \{Z_{E_a}, Z_{\Pi_a}\}$.*

Proof. Since N is nilpotent and commutes with T , we know that $\sigma(T+N) = \sigma(T)$. From the proof of [9, Theorem 3.5], we have $0 < n(T+N) \iff 0 < n(T)$ and so $E_a(T+N) = E_a(T)$. From [18, Corollary 3.8] we know that $\Pi_a(T+N) = \Pi_a(T)$. Furthermore, $\sigma_W(T+N) = \sigma_W(T)$, see [15, Lemma 2.2]. This finishes the proof. \square

Remark 3.2. We notice that the assumption of commutativity in the Proposition 3.1 is crucial.

1. Let T and N be defined on $\ell^2(\mathbb{N})$ by

$$T(x_1, x_2, \dots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \dots) \text{ and } N(x_1, x_2, \dots) = (0, \frac{-x_1}{2}, 0, 0, \dots).$$

Clearly N is nilpotent and does not commute with T . The property (Z_{E_a}) is satisfied by T , since $\sigma(T) = \{0\} = \sigma_W(T)$ and $E_a(T) = \emptyset$. But $T+N$ does not satisfy property (Z_{E_a}) as we have $\sigma(T+N) = \sigma_W(T+N) = \{0\}$ and $\{0\} = E_a(T+N)$.

2. Let T and N be defined by

$$T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots) \text{ and } N(x_1, x_2, \dots) = (0, -x_1, 0, 0, \dots).$$

N is nilpotent and $TN \neq NT$. Moreover, $\sigma(T) = \sigma_W(T) = D(0, 1)$, and $\Pi_a(T) = \emptyset$. So T satisfies property (Z_{Π_a}) . But $T+N$ does not satisfy property (Z_{Π_a}) , since $\sigma(T+N) = \sigma_W(T+N) = D(0, 1)$, and $\Pi_a(T+N) = \{0\}$.

The stability of properties (Z_{E_a}) and (Z_{Π_a}) , showed in Proposition 3.1 cannot be extended to commuting quasi-nilpotent operators, as we can see in the next Example.

Example 3.3. Let R be the operator defined on $\ell^2(\mathbb{N})$ by $R(x_1, x_2, \dots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \dots)$ and let T be the operator defined on $\ell^2(\mathbb{N})$ by $T = -R$. Clearly R is compact and quasi-nilpotent and verifies $TR = RT = -R^2$. Moreover, T satisfies properties (Z_{E_a}) and (Z_{Π_a}) , because $\sigma(T) = \{0\} = \sigma_W(T)$ and $E_a(T) = \emptyset$. But $T+R = 0$ does not satisfy neither property (Z_{E_a}) nor property (Z_{Π_a}) , since $\sigma(T+R) = \{0\} = \sigma_W(T+R)$ and $E_a(T+R) = \{0\}$, $\Pi_a(T+R) = \{0\}$. Here $\Pi^0(T+R) = \emptyset$.

However, in the next theorems, we give necessary and sufficient conditions to ensure the stability of these properties under commuting perturbations by Riesz operators which are not necessary nilpotent. The case of nilpotent operators is studied in Proposition 3.1.

Theorem 3.4. *Let $R \in \mathcal{R}(X)$ and let $T \in \mathcal{B}(X)$ which commutes with R . If T satisfies property (Z_{E_a}) , then the following statements are equivalent:*

- i) $T+R$ satisfies property (Z_{E_a}) ;
- ii) $E_a(T+R) = \Pi^0(T+R)$;
- iii) $E_a(T+R) \cap \sigma(T) \subset \Pi^0(T)$.

Proof. i) \iff ii) If $T+R$ satisfies (Z_{E_a}) , then from Lemma 2.3 we have $E_a(T+R) = \Pi^0(T+R)$. Conversely, assume that $E_a(T+R) = \Pi^0(T+R)$. Since T satisfies property (Z_{E_a}) then it satisfies Browder's theorem. From [3, Lemma 3.5], $T+R$ satisfies Browder's theorem that's $\sigma(T+R) \setminus$

$\sigma_W(T + R) = \Pi^0(T + R)$. So $T + R$ satisfies property (Z_{E_a}) .

ii) \implies iii) Assume that $\Pi^0(T + R) = E_a(T + R)$ and let $\lambda_0 \in E_a(T + R) \cap \sigma(T)$ be arbitrary. Then $\lambda_0 \in \Pi^0(T + R) \cap \sigma(T)$ and so $\lambda_0 \notin \sigma_b(T + R)$. Since we know from [17] that $\sigma_b(T) = \sigma_b(T + R)$, then $\lambda_0 \in \Pi^0(T)$. This proves that $E_a(T + R) \cap \sigma(T) \subset \Pi^0(T)$.

iii) \implies ii) Suppose that $E_a(T + R) \cap \sigma(T) \subset \Pi^0(T)$. As the inclusion $E_a(T + R) \supset \Pi^0(T + R)$ is always true, it suffices to show that $E_a(T + R) \subset \Pi^0(T + R)$. Let $\mu_0 \in E_a(T + R)$ be arbitrary. We distinguish two cases: the first is $\mu_0 \in \sigma(T)$. Then $\mu_0 \in E_a(T + R) \cap \sigma(T) \subset \Pi^0(T)$. So $\mu_0 \notin \sigma_b(T) = \sigma_b(T + R)$ and then $\mu_0 \in \Pi^0(T + R)$. The second case is $\mu_0 \notin \sigma(T)$. This implies that $\mu_0 \notin \sigma_b(T + R)$. Thus $\mu_0 \in \Pi^0(T + R)$. As a conclusion, $E_a(T + R) = \Pi^0(T + R)$. Remark that the statements ii) and iii) are always equivalent without the assumption that T satisfies property (Z_{E_a}) . \square

Similarly to Theorem 3.4, we have the following perturbation result for the property (Z_{Π_a}) .

Theorem 3.5. *Let $R \in \mathcal{R}(X)$. If $T \in \mathcal{B}(X)$ satisfies property (Z_{Π_a}) and commutes with R , then the following statements are equivalent:*

- i) $T + R$ satisfies property (Z_{Π_a}) ;
- ii) $\Pi^0(T + R) = \Pi_a(T + R)$;
- iii) $\Pi_a(T + R) \cap \sigma(T) \subset \Pi^0(T)$.

Proof. i) \iff ii) If $T + R$ satisfies (Z_{Π_a}) then from Lemma 2.9, $\Pi_a(T + R) = \Pi^0(T + R)$. Conversely, suppose that $\Pi_a(T + R) = \Pi^0(T + R)$. Since T satisfies property (Z_{Π_a}) then it satisfies Browder's theorem. Hence $T + R$ satisfies Browder's theorem that's $\sigma(T + R) \setminus \sigma_W(T + R) = \Pi^0(T + R)$. So $T + R$ satisfies property (Z_{Π_a}) .

ii) \iff iii) Goes similarly with the proof of the equivalence between the second and the third statements of Theorem 3.4. Notice also that this equivalence is always true without property (Z_{Π_a}) for T . \square

The following example proves in general that, the properties (Z_{E_a}) and (Z_{Π_a}) are not preserved under commuting finite rank power perturbations.

Example 3.6. For fixed $0 < \varepsilon < 1$, let F_ε be a finite rank operator defined on $\ell^2(\mathbb{N})$ by $F_\varepsilon(x_1, x_2, x_3, \dots) = (-\varepsilon x_1, 0, 0, \dots)$. We consider the operators T and F defined by $T = R \oplus I$ and $F = 0 \oplus F_\varepsilon$. F is a finite rank operator and $TF = FT$. We have,

$$\sigma(T) = \sigma(R) \cup \sigma(I) = D(0, 1), \sigma_a(T) = \sigma_a(R) \cup \sigma_a(I) = C(0, 1), \sigma_W(T) = D(0, 1),$$

$$\sigma(T + F) = \sigma(R) \cup \sigma(I + F_\varepsilon) = D(0, 1), \sigma_W(T + F) = D(0, 1) \text{ and}$$

$$\sigma_a(T + F) = \sigma_a(R) \cup \sigma_a(I + F_\varepsilon) = C(0, 1) \cup \{1 - \varepsilon\}.$$

Moreover, $E_a(T) = \Pi_a(T) = \emptyset$. So T satisfies properties (Z_{E_a}) and (Z_{Π_a}) . But, since $E_a(T + F) = \Pi_a(T + F) = \{1 - \varepsilon\}$, then $T + F$ does not satisfy either property (Z_{E_a}) nor property (Z_{Π_a}) . Here $\Pi_a(T + F) \cap \sigma(T) = \{1 - \varepsilon\}$, $\Pi^0(T) = \Pi^0(T + F) = \emptyset$.

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